Logic

So far we have described truth conditions through translation to English, trying to be precise.

Today: Propositional logic, connectives, predicate logic with variables

1 Propositional logic[1]

Propositional logic gives us a way to precisely describe *relationships of meaning* between individual propositions.

(1) a. Cheryl ate the noodles.
    b. Cheryl ate the salad and the noodles.

(2) a. If tomorrow is a public holiday, there will be no class tomorrow.
    b. Tomorrow is a public holiday.
    c. There will be no class tomorrow.

Propositional logic can be thought of as its own toy language, with a toy syntax and a toy semantics. We will call "grammatical sentences" in propositional logic *well-formed formulas (wff)*.

(3) **The syntax of propositional logic:**
  a. Atomic formulas: $p, q, r, ...$[2] are all wffs.
     The atomic formulas function as the "lexicon" for this "language."
  b. Negation: If φ is a wff, $\neg\varphi$ is also a wff.
  c. Binary connectives: if φ and ψ are wffs, then so are:

     i. $[\varphi \wedge \psi]$ ii. $[\varphi \vee \psi]$ iii. $[\varphi \rightarrow \psi]$ iv. $[\varphi \leftrightarrow \psi]$

  d. Nothing else is a wff.

I'll be sloppy and drop the [ ] except where they're necessary to make the "constituency" clear.

[1]Much of today's handout is inspired by materials by Masha Esipova & Lucas Champollion and Liz Coppock.
[2]*IFS* uses capital letters P, Q, R, etc. But mathematics is invariant under change of notation.

Exercise: * the formulas in (4) which are not wffs:

(4)

a.	$p \wedge p$	d.	$\neg p \vee q$	g.	$q\neg p$
b.	$\neg\neg p$	e.	$pq\wedge$	h.	$q \rightarrow q$
c.	$\neg \wedge p$	f.	$p\neg$	i.	$q \leftarrow p$

We then need a way to interpret wffs:

- Each atomic proposition is either true or false in a given model: e.g. $[\![p]\!]^M = 1$ or $[\![q]\!]^M = 0$
- We then define a semantics for each of the operators in (3b–c) above, which we can express using *truth tables*.

Negation ¬: "not"

In any model M, $[\![\neg\varphi]\!]^M = \begin{cases} 0 & \text{if } [\![\varphi]\!]^M = 1 \\ 1 & \text{if } [\![\varphi]\!]^M = 0 \end{cases}$

p	$\neg p$
1	
0	

Conjunction ∧: "and"

p	q	$p \wedge q$
1	1	
1	0	
0	1	
0	0	

Exercise: (5) is ambiguous. Describe the two readings in terms of p = "Keith had beer" and q = "Keith had durian."

(5) Keith didn't have beer and durian.

a.

b.

What is the relationship between (5a) and (5b)? Use the truth table to help guide your thinking.

p	q	$p \wedge q$	$\neg p$	$\neg q$	(5a)	(5b)
1	1					
1	0					
0	1					
0	0					

Disjunction $\vee$: We want this (roughly) to correspond to natural language "or," but here there's a challenge. Consider:

(6) We can meet today or tomorrow.

(7) Your taxes will be lower if you are over 65 or blind.

We define $\vee$ as inclusive disjunction and refer to exclusive disjunction as XOR:

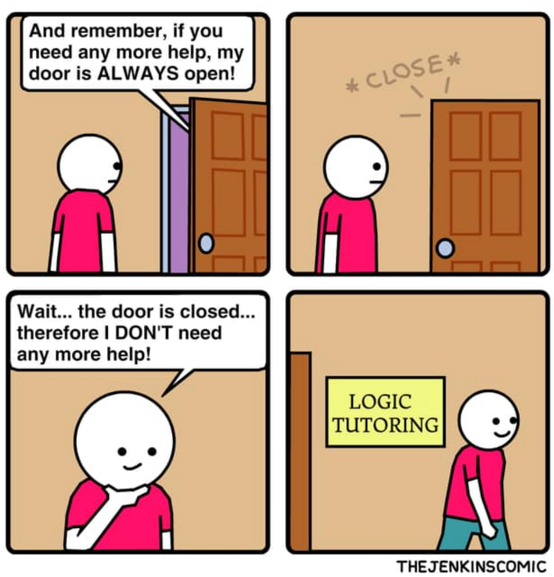
p	q	$p \vee q$
1	1	
1	0	
0	1	
0	0	

p	q	p XOR q
1	1	
1	0	
0	1	
0	0	

Conditional $\rightarrow$ and biconditional $\leftrightarrow$: We want $\rightarrow$ to (very roughly) correspond to "if...then," but the behavior of natural language conditionals turns out to be quite complicated. The meaning adopted for $\rightarrow$ is traditionally called the *material conditional*.

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1



See discussion of conditionals in *IFS* §3, especially §3.2.3.

(8) **De Morgan's laws:**

a. $\neg p \wedge \neg q$ iff $\neg[p \vee q]$

b. $\neg p \vee \neg q$ iff $\neg[p \wedge q]$

Exercise: Prove (8b) using a truth table:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg[p \wedge q]$
1	1					
1	0					
0	1					
0	0					

Q: Can we really "prove" something by truth tables?

A: *Yes!* if done correctly. Each row in a truth table corresponds to a particular model (or class of models). By checking each possible combination of truth values for the atomic formulas, we are essentially checking across all possible models.[3]

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(9) **Summary: Semantics for propositional logic**

a. Atomic formulas: Each atomic formula is true or false; e.g. $[\![p]\!]^M = 1$ or $[\![p]\!]^M = 0$.

b. Negation: $[\![\neg\varphi]\!]^M = 1$ iff $[\![\varphi]\!]^M = 0$

c. Binary connectives:

$[\![[\varphi \wedge \psi]]\!]^M = 1$ iff $[\![\varphi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$

$[\![[\varphi \rightarrow \psi]]\!]^M = 1$ iff $[\![\varphi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$

$[\![[\varphi \vee \psi]]\!]^M = 1$ iff $[\![\varphi]\!]^M = 1$ or $[\![\psi]\!]^M = 1$

$[\![[\varphi \leftrightarrow \psi]]\!]^M = 1$ iff $[\![\varphi]\!]^M = [\![\psi]\!]^M$

[3]If the relevant formulas depend on n atomic propositions, there will be 2^n rows to check. Here, with just p and q, there are $2^2 = 4$ rows to check.

2 Predicate logic with variables[4]

We now introduce a way of talking about predicates and their arguments, not just relationships between wffs (i.e. sentences).

(10) **Individuals and predicates:**

a. Individuals include John, Mary, Taro, Hanako, Pochi, Fido, etc.

b. One-place (unary) predicates: Student, Japanese, Smart, Barks, Dog, Cat, etc.
If π is a one-place predicate and α is an individual, $\pi(\alpha)$ is an atomic formula.

c. Two-place (binary) predicates: Likes, Petted, Kissed, etc.
If π is a two-place predicate and α, β are individuals, $\pi(\alpha, \beta)$ is an atomic formula.

Some examples of wff:

(11) Student(John)

(12) ¬¬Japanese(Pochi)

(13) Petted(Mary,Pochi) ↔ Likes(Mary,Pochi)

(14) Happy(Fido) → Barks(Fido)

To this, we can add *variables*:

(15) **Variables and two quantifiers:**

a. Variables: x, y, z etc. These variables stand in for individuals.

b. Quantification: If φ is a wff and u is a variable, $[\forall u \,.\, \varphi]$ and $[\exists u \,.\, \varphi]$ are wffs.
They are interpreted as follows:

i. $[\![\forall u \,.\, \varphi]\!]^M = 1$ iff for all individuals k, $[\![\varphi]\!]^M = 1$ when $u = k$

ii. $[\![\exists u \,.\, \varphi]\!]^M = 1$ iff there is at least one individual k such that $[\![\varphi]\!]^M = 1$ when $u = k$

(16) $\forall x \,.\, [\text{Student}(x) \rightarrow \text{Smart}(x)]$

(17) $\exists y \,.\, [\text{Dog}(y) \wedge \text{Smart}(y)]$

[4] My presentation here is somewhat informal, to try to keep things very digestable. *IFS* includes a much more formal presentation of predicate logic with variables, in the section beyond what is assigned.

Notes on variables

(18) **Some math "sentences":**

a. $1 = 2 - 1$ a sentence with no variables; not context-sensitive

b. $n = 2 - 1$ a sentence with a variable; context-sensitive

c. $\forall n\ (2(n+1) = 2n + 2)$ a sentence with a variable; *not* context-sensitive

- We say (18b) contains a *free variable* because the truth of the sentence depends on the context. In particular, the sentence is true iff the variable "n" is interpreted as 1.
- The truth of sentence (18c), like (18a), does not depend on the context at all.

(19) **Some terminology, using (18c) as an example:**

$$\underset{binder}{\forall n}\ \underbrace{\left(2(\underset{bound}{n}+1) = 2\underset{bound}{n}+2\right)}_{scope}$$

- *Binders* control the interpretation of a particular variable within a certain part of its structure, which we call its *scope*. Here, $\forall$ *binds* the variable n in its scope.
- We call variables that are in the scope of a matching binder *bound variables*.

More examples:

(20) $\exists y\ .\ [\text{Green}(y)]$

(21) $\exists x\ .\ [\text{Like}(x, \text{Taro})]$

(22) $\{x : \text{Like}(x, \text{Taro})\}$

Just as we saw with {... : ...} notation before, we can write some weird meanings with $\exists$ and $\forall$:

(23) $\forall y\ .\ [\text{Dog}(\text{Fido})]$ *vacuous quantification*

Note that bound variables are interchangeable, if you carefully substitute the binder and bound variables together:

(24) $\exists y\ .\ [\text{Green}(y) \wedge \text{Dog}(y)]\ =\ \exists z\ .\ [\text{Green}(z) \wedge \text{Dog}(z)]$

(25) $\{x : \text{Like}(x, y)\}\ \neq\ \{y : \text{Like}(x, y)\}$