Logic

Announcements: office hours: Thursdays 11:30–noon; submission in PDF please

So far we have described truth conditions through translation to English, trying to be precise. But we can do better than this, using the tools of mathematical logic.
Today: Propositional logic, connectives, predicate logic

1 A note on conservativity

But first, from the problem set...

(1) A quantificational determiner Q is called *conservative* if $Q(A)(B)$ is true if and only if $Q(A)(A \cap B)$ is true.

► **Claim (Barwise and Cooper, 1981):** All quantificational determiners in natural language are conservative.

An example of a non-conservative quantifier meaning is *allnon* from the problem set.

- Hunter and Lidz 2012: "We present experimental evidence that 4- and 5-year-olds fail to learn a novel non-conservative determiner but succeed in learning a comparable conservative determiner."

Warm up: Consider *only* as in sentences like "Only dogs bark." Define the truth conditions of *only*$(A)(B)$ in terms of the sets A and B. Show that this *only* is not conservative.

2 Propositional logic[1]

Propositional logic gives us a way to precisely describe *relationships of meaning* between individual propositions.

(2) a. Cheryl ate the noodles.
b. Cheryl ate the salad and the noodles.

(3) a. If tomorrow is a public holiday, there will be no class tomorrow.
b. Tomorrow is a public holiday.
c. There will be no class tomorrow.

[1]Much of today's handout is inspired by materials by Masha Esipova & Lucas Champollion and Liz Coppock.

Propositional logic can be thought of as its own toy language, with a toy syntax and a toy semantics. We will call "grammatical sentences" in predicate logic *well-formed formulas (wff)*.

(4) **The syntax of propositional logic:**

a. Atomic formulas: $p, q, r, \ldots$[2] are all wffs.
   The atomic formulas function as the "lexicon" for this "language."

b. Negation: If ϕ is a wff, $\neg\phi$ is also a wff.

c. Binary connectives: if ϕ and ψ are wffs, then so are:

   i. $[\phi \wedge \psi]$ ii. $[\phi \vee \psi]$ iii. $[\phi \to \psi]$ iv. $[\phi \leftrightarrow \psi]$

d. Nothing else is a wff.

I'll be sloppy and drop the [ ] except in cases where they're necessary to make the "constituency" clear.

Exercise: * the formulas in (5) which are not wffs:

(5)

a.	$p \wedge p$	d.	$\neg p \vee q$	g.	* $q\neg p$
b.	$\neg\neg p$	e.	* $pq\wedge$	h.	$q \to q$
c.	* $\neg \wedge p$	f.	* $p\neg$	i.	* $q \leftarrow p$

We then need a way to interpret wffs:

- Each atomic proposition is either true or false in a given model: e.g. $[\![p]\!]^M = 1$ or $[\![q]\!]^M = 0$
- We then define a semantics for each of the operators in (18b–c) above, which we can express using *truth tables*.

Negation $\neg$: "not"

In any model M, $[\![\neg\phi]\!]^M = \begin{cases} 0 & \text{if } [\![\phi]\!]^M = 1 \\ 1 & \text{if } [\![\phi]\!]^M = 0 \end{cases}$

p	$\neg p$
1	0
0	1

Conjunction $\wedge$: "and"

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

[2] *IFS* uses capital letters P, Q, R, etc. But mathematics is invariant under change of notation.

Exercise: (6) is ambiguous. Describe the two readings in terms of p = "Keith had beer" and q = "Keith had durian."

(6) Keith didn't have beer and durian.

a. $\neg[p \wedge q]$

b. $\neg p \wedge \neg q$

What is the relationship between (6a) and (6b)? Use the truth table to help guide your thinking.

p	q	$p \wedge q$	$\neg p$	$\neg q$	(6a)	(6b)
1	1					
1	0					
0	1					
0	0					

Disjunction $\vee$: We want this (roughly) to correspond to natural language "or," but here there's a challenge. Consider:

(7) We can meet today or tomorrow. (probably) not both = *exclusive*

(8) Your taxes will be lower if you are over 65 or blind. lower if both = *inclusive*

We define $\vee$ as inclusive disjunction and refer to exclusive disjunction as XOR:

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

p	q	p XOR q
1	1	0
1	0	1
0	1	1
0	0	0

Exercise: Complete the truth table for (9). Since (9) appears to be the negation of (7), what does this suggest about English "or"?

(9) We can't meet today or tomorrow.

p	q	(9)
1	1	
1	0	
0	1	
0	0	

Conditional $\rightarrow$ and biconditional $\leftrightarrow$: We want $\rightarrow$ to (very roughly) correspond to "if...then," but the behavior of natural language conditionals turns out to be quite complicated. The meaning adopted for $\rightarrow$ is traditionally called the *material conditional*.

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

See discussion of conditionals in *IFS* §3, especially §3.2.3.

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(10) **De Morgan's laws:**

a. $\neg p \wedge \neg q$ iff $\neg[p \vee q]$

b. $\neg p \vee \neg q$ iff $\neg[p \wedge q]$

Exercise: Prove (10b) using a truth table:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg[p \wedge q]$
1	1					
1	0					
0	1					
0	0					

Q: Can we really "prove" something by truth tables?

A: *Yes!* if done correctly. Each row in a truth table corresponds to a particular model (or class of models). By checking each possible combination of truth values for the atomic formulas, we are essentially checking across all possible models.[3]

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(11) **Summary: Semantics for propositional logic**

a. Atomic formulas: Each atomic formula is true or false; e.g. $[\![p]\!]^M = 1$ or $[\![p]\!]^M = 0$.

b. Negation: $[\![\neg\phi]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$

c. Binary connectives:

$[\![[\phi \wedge \psi]]\!]^M = 1$ iff $[\![\phi]\!]^M = 1$ and $[\![\phi]\!]^M = 1$ $\quad$ $[\![[\phi \rightarrow \psi]]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$ or $[\![\phi]\!]^M = 1$

$[\![[\phi \vee \psi]]\!]^M = 1$ iff $[\![\phi]\!]^M = 1$ or $[\![\phi]\!]^M = 1$ $\quad$ $[\![[\phi \leftrightarrow \psi]]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\phi]\!]^M$

[3]If the relevant formulas depend on n atomic propositions, there will be 2^n rows to check. Here, with just p and q, there are $2^2 = 4$ rows to check.

3 Predicate logic

We now introduce a way of talking about predicates and their arguments, not just relationships between wffs (i.e. sentences).

(12) **The syntax of predicate logic without variables:**

a. The lexicon:

i. *Individual constants*: j, m, t, h, p, f, etc. Individual constants are *terms*; nothing else is a term.

ii. One-place (unary) *predicates*: Student, Japanese, Smart, Barks, Dog, Cat, etc.

iii. Two-place (binary) predicates: Likes, Petted, Kissed, etc.

b. Atomic formulas:

i. If π is a one-place predicate and α is a term, then $\pi(\alpha)$ is a wff.

ii. If π is a two-place predicate and α and β are terms, then $\pi(\alpha, \beta)$ is a wff.

c. Equality: If α and β are terms, $[\alpha = \beta]$ is a wff.

d. Negation: If ϕ is a wff, $\neg\phi$ is also a wff.

e. Binary connectives: if ϕ and ψ are wffs, then so are:

i. $[\phi \wedge \psi]$ ii. $[\phi \vee \psi]$ iii. $[\phi \rightarrow \psi]$ iv. $[\phi \leftrightarrow \psi]$

f. Nothing else is a wff.

Parts (d–f) should look familiar! We also need semantics for the unfamiliar parts:

(13) **Additional semantics for predicate logic:** (see *IFS* warning below)

a. Individual constants are interpreted as individuals in the domain D: for example, $[\![\text{j}]\!]^M$ = John, $[\![\text{m}]\!]^M$ = Mary, $[\![\text{t}]\!]^M$ = Taro, $[\![\text{h}]\!]^M$ = Hanako, $[\![\text{p}]\!]^M$ = Pochi, $[\![\text{f}]\!]^M$ = Fido

b. $[\![\alpha = \beta]\!]^M = 1$ iff $[\![\alpha]\!]^M = [\![\beta]\!]^M$

c. If π is a one-place predicate, $[\![\pi(\alpha)]\!]^M = 1$ iff $[\![\alpha]\!]^M \in [\![\pi]\!]^M$ (so we think of π as a set)

d. If π is a two-place predicate, $[\![\pi(\alpha, \beta)]\!]^M = 1$ iff $\langle [\![\alpha]\!]^M, [\![\beta]\!]^M \rangle \in [\![\pi]\!]^M$ (so we think of π as a set of ordered pairs)

(14) Student(j)

(15) ¬¬Japanese(p)

(16) Petted(m,p) ↔ Likes(m,p)

(17) Dog(h) → Barks(h)

Note/warning on *IFS*: *IFS* draws a distinction between the "denotation function" for model M, $[\![\cdot]\!]^M$, and the model's "interpretation function" I. Here, I'm skipping the I step. The theory in the book is an "indirect" interpretation theory, whereas the theory here is a "direct" interpretation theory. See discussion in *IFS* §3.4.1.

IFS also adds "complex terms" with functions to this; see *IFS* §3.4.3.

What we're building up to is *predicate logic with variables*, which additionally allows us to discuss some basic forms of quantification:

(18) **The syntax of predicate logic without variables:**

a. The lexicon:

i. Individual constants: j, m, t, h, p, f, etc. Individual constants are terms.

ii. Variables: x, y, z, etc. Variables are also terms. Nothing else is a term.

iii. One-place (unary) *predicates*: Student, Japanese, Smart, Barks, Dog, Cat, etc.

iv. Two-place (binary) predicates: Likes, Petted, Kissed, etc.

b. Atomic formulas: ...

c. Quantification: If ϕ is a wff and u is a variable, $[\forall u \,.\, \phi]$ and $[\exists u \,.\, \phi]$ are wffs.

d. Equality: ... e. Negation: ... f. Binary connectives: ...

g. Nothing else is a wff.

(19) $\forall x \,.\, [\text{Student}(x) \rightarrow \text{Smart}(x)]$

(20) $\exists y \,.\, [\text{Dog}(y) \wedge \text{Smart}(y)]$

(21) **Additional semantics for predicate logic with variables:**

We now additionally keep track of an *assignment function* g, which maps variables to

a. For u a variable, $[\![u]\!]^{M,g} = g(u)$

b. $[\![\forall u \,.\, \phi]\!]^{M,g} = 1$ iff for all individuals $k \in D$, $[\![\phi]\!]^{M,[u \mapsto k] || g} = 1$

c. $[\![\exists u \,.\, \phi]\!]^{M,g} = 1$ iff there is an individual $k \in D$ such that $[\![\phi]\!]^{M,[u \mapsto k] || g} = 1$

The function $[u \mapsto k] \,||\, g$ maps u to k but otherwise behaves like g. (*IFS*: $g[u \mapsto k]$)

Some special kinds of meanings:

(22) $\text{Dog}(z)$ *open formulas* are assignment-dependent

(23) $\exists x \,.\, \text{Dog}(p)$ *vacuous quantification*

References

Barwise, Jon, and Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4:159–219.

Hunter, Tim, and Jeffrey Lidz. 2012. Conservativity and learnability of determiners. *Journal of Semantics* 30:315–334.