Sets, functions, quantifiers

1 Set notation and concepts

$\{0,1\}$	a *set* with two *members/elements*, 0 and 1
	We will often use 0 and 1 for the truth values, *false* and *true*.
$\{0,1\} = \{1,0\}$	Members of sets are unordered.
D_e	the *domain* of all individuals (in the model); sometimes also called the *universe*
$\emptyset$	the empty set, $\{\}$
$\{x \mid x \text{ is a cat}\}$	the set which contains everything in D_e that is a cat
	Could be read "the set of x such that x is a cat."[1]
$x \in A$	x is a member of the set A
	Note that $a \in \{\{a\}, b\}$ is false.

We generally use uppercase letters for sets and lowercase letters for individuals in D_e.

(1) Tama $\in \{x \mid x \text{ is a cat}\}$

(2) $\{x \mid x \text{ saw a picture of } x\}$

(3) $\{x \mid \text{this is semantics class}\}$

$A \cap B$	the *intersection* of A and B: $\{x \mid x \in A \text{ and } x \in B\}$
$A \cup B$	the *union* of A and B: $\{x \mid x \in A \text{ or } x \in B\}$
$B - A$	the *complement* of A in B: $\{x \mid x \in B \text{ and } x \notin A\}$[2]
$\mid A \mid$	the *cardinality/size* of A; for example $\mid \{0,1\} \mid = 2$

(4) a. Tama $\in \{x \mid x \text{ is a cat}\} \cap \{y \mid y \text{ is black}\}$
 b. Tama $\in \{x \mid x \text{ is a cat and } y \text{ is black}\}$
 c. Tama $\in \{x \mid x \text{ is a cat and } x \text{ is black}\}$

$A \subseteq B$	A is a *subset* of B; every member of A is in B.
$A = B$	$A \subseteq B$ and $B \subseteq A$

The sets above are all subsets of D_e, but we can also write sets of other things:

(5) $\{A \mid A \subseteq \{x \mid x \text{ is a cat}\}\}$

What is in this set?

[1] *IFS* calls this "predicate notation." Note that some people use a colon and write $\{x : x \text{ is a cat}\}$.
[2] Some people write $B \setminus A$ instead.

Exercise (from Heim & Kratzer pages 9–10)

Consider the following equations. Are they always true, never true, or true under specific circumstances? Under what circumstances?

(6) a. $\{a\} = \{b\}$

true when $a = b$

b. $\{x \mid x = a\} = \{a\}$

always true

c. $\{x \mid x \text{ is green}\} = \{y \mid y \text{ is green}\}$

always true

d. $\{x \mid x \text{ likes } a\} = \{y \mid y \text{ likes } b\}$

when the people/things that like a are the same set as the people/things that like b

e. $\{x \mid x \in A\} = A$

always true

Some other notation, useful in proofs:

$\Rightarrow$	entails
$\Longleftrightarrow$	if and only if (iff)

Exercises

(7) Prove that, for any sets A and B, $A - B \subseteq A$. When is $A - B = A$?

Proof: Suppose $x \in A - B \Longleftrightarrow x \in A$ and $x \notin B$

$\Longrightarrow x \in A$

Equality holds when A and B do not overlap: $A \cap B = \emptyset$

(8) Prove that, for any sets A and B, $A \cap B \subseteq A \cup B$. When is $A \cap B = A \cup B$?

Proof: Suppose $x \in A \cap B \Longleftrightarrow x \in \{y \mid y \in A \text{ and } y \in B\}$

$\Longleftrightarrow x \in A$ and $x \in B$

$\Longrightarrow x \in A$ or $x \in B$

$\Longleftrightarrow x \in A \cup B$

Hints for proofs with sets:

- First, choose some arbitrary but concrete sets A and B and see if the statements are true and make sense to you.
- Make sure you consider $A = \emptyset$ and $B = \emptyset$ too.
- To prove that $A \subseteq B$, show that any member of A is necessarily also a member of B.
- To prove that $A = B$, show that $A \subseteq B$ and $B \subseteq A$.

2 Predicates as sets

(9) a. Tama is a cat.

b. Tama is a black cat.

What is the meaning relationship between (9a) and (9b)? Is it an entailment or presupposition?

(10) a. $A = [\![\text{cat}]\!] = \{x \mid x \text{ is a cat}\}$

b. $B = [\![\text{black cat}]\!] = \{x \mid x \text{ is black and } x \text{ is a cat}\}$

What is the relationship between A and B?

Exercises

What is the relationship between the (a) and (b) sets?

(11) a. $\{x \mid x \text{ has read two books}\}$

b. $\{x \mid x \text{ has read three books}\}$

(12) a. $\{x \mid x \text{ is a student that has read two books}\}$

b. $\{x \mid x \text{ is a student that has read three books}\}$

3 Predicates as functions

A *function* takes an element from one set (its *domain*) and returns an element from another set (its *range*). What are the domain and ranges for the following functions?

(13) $f(x) = x + 3$

(14) $g(x) = \begin{bmatrix} \text{Tama} & \rightarrow & 1 \\ \text{Mitsi} & \rightarrow & 1 \\ \text{Pochi} & \rightarrow & 0 \\ \text{Fido} & \rightarrow & 0 \end{bmatrix}$

(15) $h(x) = x$'s mother

Predicates can also be expressed as functions:

(16) $[\![\text{cat}]\!]\ (x) =$

It will sometimes be useful to think of predicates as sets, and sometimes useful to think of predicates as functions, so we will go back and forth. Predicates as functions can be thought of as *characteristic functions*, which take an argument and check whether :

(17) The characteristic function for set A:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

(14) is a characteristic function for $[\![\text{cat}]\!]^M$, for a model M with just those four individuals in D_e.

4 Quantificational determiners

Consider quantifiers in subject position. What are the truth conditions of these sentences?

(18) Every cat sleeps.

(19) Some cat sleeps.

Following an influential approach called *Generalized Quantifier Theory* (Barwise and Cooper, 1981), we will think about the meaning of quantifiers as relations between sets.

(20) $[\![\text{Every cat sleeps}]\!]^M = 1$ iff $[\![\text{cat}]\!]^M \subseteq [\![\text{sleep}]\!]^M$

(21) $[\![\text{Some cat sleeps}]\!]^M = 1$ iff $[\![\text{cat}]\!]^M \cap [\![\text{sleep}]\!]^M \neq \emptyset$

Based on these results, we can define the general functions *every* and *some*:

(22) $every(A)(B) = 1$ iff $A \subseteq B$ returns the truth value for "Every A is B."

(23) $some(A)(B) = 1$ iff $A \cap B \neq \emptyset$ returns the truth value for "Some A is B."

Exercise: Take a quantifier in a sentence of the form "Quantifier A B." For example, in "Every student is sleeping," the quantifier is "every," A = "student," and B = "is sleeping." Define the truth conditions of "Quantifier A B" in terms of the sets A and B. (We ignore number morphology and agreement here.)

Answers:

(24) $every(A)(B) = 1$ iff $A \subseteq B$

(25) $a/some(A)(B) = 1$ iff $A \cap B \neq \emptyset$

(26) $no(A)(B) = 1$ iff $A \cap B = \emptyset$

(27) $two(A)(B) = 1$ iff $|A \cap B| \geq 2$

(28) $more\text{-}than\text{-}two(A)(B) = 1$ iff $|A \cap B| > 2$

(29) $most(A)(B) = 1$ iff $|A \cap B| > |A - B|$

(30) $not\ all(A)(B) = 1$ iff $A - B \neq \emptyset$

(31) $the(A)(B) = 1$ iff $|A| = 1$ and $A \subseteq B$

5 Downward entailment and NPIs

What is the meaning relationship between these pairs, which vary in the underlined positions?

(32) a. Stephanie read two books.
b. Stephanie read three books.

(33) a. I know that S. reads two books.
b. I know that S. reads three books.

(34) a. I doubt that Stephanie read two books.
b. I doubt that Stephanie read three books.

(35) a. If S. reads two books, she will pass.
b. If S. reads three books, she will pass.

(36) A function f is *downward-entailing* (DE; or downward monotone) if and only if for all $A \subseteq B \subseteq D_e$, $f(B) \Rightarrow f(A)$.

Now consider noun phrases like *any books*, *anyone*, *anything*. These are called *negative polarity items* (NPIs). Contrasts like in (46) make it look like NPIs are sensitive to negation:

(37) a. * Stephanie read *any* book(s).
b. * I know that Stephanie read *any* book(s).
c. I doubt that Stephanie read *any* book(s).
d. If Stephanie read *any* book(s), she will pass.

(38) NPIs are allowed in *downward-entailing* environments. (Ladusaw, 1979)

For quantificational determiners, we have to be clear whether we're looking at the first argument or second argument.

(39) a. Every cat sleeps.
b. Every black cat sleeps.

(40) a. No cat sleeps.
b. No black cat sleeps.

(41) a. Some cat sleeps.
b. Some black cat sleeps.

(42) a. Every student has read two books.
b. Every student has read three books.

(43) a. No student has read two books.
b. No student has read three books.

(44) a. Some student has read two books.
b. Some student has read three books.

(45) a. Every student that has read *any* book(s) will pass.
b. No student that has read *any* book(s) will pass.
c. * Some student that has read *any* book(s) will pass.

(46) a. * Every student has read *any* book(s).
b. No student has read *any* book(s).
c. * Some student has read *any* book(s).

References

Barwise, Jon, and Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4:159–219.

Ladusaw, William A. 1979. Polarity sensitivity as inherent scope relations. Doctoral Dissertation, University of Texas at Austin.