Sets, quantifiers, and entailment

1 Set notation and concepts

$\{0,1\}$	a *set* with two *members/elements*, 0 and 1
	We will often use 0 and 1 for the truth values, *false* and *true*.
$\{0,1\} = \{1,0\}$	Members of sets are unordered.
D_e	the *domain* of all individuals (in the model); sometimes also called the *universe*
$\emptyset$	the empty set, $\{\}$
$\{x : x \text{ is a cat}\}$	the set which contains everything in D_e that is a cat
	Could be read "the set of x such that x is a cat."
$x \in A$	x is a member of the set A
	Note that $a \in \{\{a\}, b\}$ is false.

We generally use uppercase letters for sets and lowercase letters for individuals in D_e.

(1) Tama $\in \{x : x \text{ is a cat}\}$

(2) $\{x : x \text{ saw a picture of } x\}$

(3) $\{x : \text{this is semantics class}\}$

$A \cap B$	the *intersection* of A and B: $\{x : x \in A \text{ and } x \in B\}$
$A \cup B$	the *union* of A and B: $\{x : x \in A \text{ or } x \in B\}$
$B \setminus A$	the *complement* of A in B: $\{x : x \in B \text{ and } x \notin A\}$
$\|A\|$	the *cardinality/size* of A; for example $\|\{0,1\}\| = 2$

(4) a. Tama $\in \{x : x \text{ is a cat}\} \cap \{y : y \text{ is black}\}$
  b. Tama $\in \{x : x \text{ is a cat and } y \text{ is black}\}$
  c. Tama $\in \{x : x \text{ is a cat and } x \text{ is black}\}$

$A \subseteq B$	A is a *subset* of B; every member of A is in B.
$A = B$	$A \subseteq B$ and $B \subseteq A$

The sets above are all subsets of D_e, but we can also write sets of other things:

(5) $\{A : A \subseteq \{x : x \text{ is a cat}\}\}$

What is in this set?

Some other notation, useful in proofs:

$\Rightarrow$	entails
$\iff$	if and only if (iff)

Exercises

(6) Prove that, for any sets A and B, $A \setminus B \subseteq A$. When is $A \setminus B = A$?

Proof: Suppose $x \in A \setminus B \iff x \in A$ and $x \notin B$

$\implies x \in A$

Equality holds when A and B do not overlap: $A \cap B = \emptyset$

(7) Prove that, for any sets A and B, $A \cap B \subseteq A \cup B$. When is $A \cap B = A \cup B$?

Proof: Suppose $x \in A \cap B \iff x \in \{y : y \in A \text{ and } y \in B\}$

$\iff x \in A$ and $x \in B$

$\implies x \in A$ or $x \in B$

$\iff x \in A \cup B$

Hints for proofs with sets:

- First, choose some arbitrary but concrete sets A and B and see if the statements are true and make sense to you.
- Make sure you consider $A = \emptyset$ and $B = \emptyset$ too.
- To prove that $A \subseteq B$, show that any member of A is necessarily also a member of B.
- To prove that $A = B$, show that $A \subseteq B$ and $B \subseteq A$.

2 Predicates as sets

(8) a. Tama is a cat.

b. Tama is a black cat.

What is the meaning relationship between (8a) and (8b)? Is it an entailment or presupposition?

(9) a. $A = [\![\text{cat}]\!] = \{x : x \text{ is a cat}\}$

b. $B = [\![\text{black cat}]\!] = \{x : x \text{ is black and } x \text{ is a cat}\}$

What is the relationship between A and B?

NB: Winter's Truth Conditionality Criterion ensures the parallel between (8) and (9).

The TCC requires that S_1 intuitively entail S_2 iff, for any model M, $[\![S_1]\!]^M \leq [\![S_2]\!]^M$. Let the truth conditions of (8a,b) be:

– $[\![\text{Tama is a cat (8a)}]\!]^M = 1$ iff Tama $\in A$ in M

– $[\![\text{Tama is a black cat (8b)}]\!]^M = 1$ iff Tama $\in B$ in M

The fact that (8b) $\Rightarrow$ (8a) means that, for any model, if Tama $\in B$, Tama $\in A$. But this is equivalent to saying that in any model, $B \subseteq A$.

Exercises

What is the relationship between the (a) and (b) sets?

(10) a. $\{x : x \text{ has read three books}\}$

b. $\{x : x \text{ has read two books}\}$

(11) a. $\{x : x \text{ is a student that has read three books}\}$

b. $\{x : x \text{ is a student that has read two books}\}$

3 Quantificational determiners

(12) Every cat sleeps.

(13) Some cat sleeps.

What are the truth conditions of these sentences?

Following an influential approach called *Generalized Quantifier Theory* (Barwise and Cooper, 1981), we will think about the meaning of quantifiers as relations between sets.

(14) $[\![\text{Every cat sleeps}]\!]^M = 1 \text{ iff } [\![\text{cat}]\!]^M \subseteq [\![\text{sleep}]\!]^M$

(15) $[\![\text{Some cat sleeps}]\!]^M = 1 \text{ iff } [\![\text{cat}]\!]^M \cap [\![\text{sleep}]\!]^M \neq \emptyset$

Based on these results, we can define the general functions *every* and *some*:

(16) $every(A)(B) = 1 \text{ iff } A \subseteq B$ returns the truth value for "Every A is B."

(17) $some(A)(B) = 1 \text{ iff } A \cap B \neq \emptyset$ returns the truth value for "Some A is B."

Exercise: Take a quantifier in a sentence of the form "Quantifier A B." For example, in "Every student is sleeping," the quantifier is "every," A = "student," and B = "is sleeping." Define the truth conditions of "Quantifier A B" in terms of the sets A and B. (We ignore number morphology and agreement here.)

Answers:

(18) $every(A)(B) = 1$ iff $A \subseteq B$

(19) $a/some(A)(B) = 1$ iff $A \cap B \neq \emptyset$

(20) $no(A)(B) = 1$ iff $A \cap B = \emptyset$

(21) $two(A)(B) = 1$ iff $|A \cap B| = 2$

(22) $more\text{-}than\text{-}two(A)(B) = 1$ iff $|A \cap B| > 2$

(23) $most(A)(B) = 1$ iff $|A \cap B| > |A \setminus B|$

(24) $not\ all(A)(B) = 1$ iff $A \setminus B \neq \emptyset$

(25) $the(A)(B) = 1$ iff $|A| = 1$ and $A \subseteq B$

4 Downward entailment and NPIs

What is the meaning relationship between these pairs?

(26) a. Every cat sleeps.
b. Every black cat sleeps.

(27) a. No cat sleeps.
b. No black cat sleeps.

(28) a. Some cat sleeps.
b. Some black cat sleeps.

(29) A quantificational determiner D is *left downward-entailing* (DE; or downward monotone) if and only if for all $A_1 \subseteq A_2 \subseteq D_e$ and $B \subseteq D_e$, $D(A_2, B) \Rightarrow D(A_1, B)$.

Every and *no* are left DE. "Downward" because we can replace a set with a subset (down) and the result will be entailed. (*Some* is left upward-entailing.)

(30) a. Every student has read three books.
b. Every student has read two books.

(31) a. No student has read three books.
b. No student has read two books.

(32) a. Some student has read three books.
b. Some student has read two books.

(33) A quantificational determiner D is *right downward-entailing* (DE; or downward monotone) if and only if for all $B_1 \subseteq B_2 \subseteq D_e$ and $A \subseteq D_e$, $D(A, B_2) \Rightarrow D(A, B_1)$.

Now consider noun phrases like *any books*, *anyone*, *anything*. These are called *negative polarity items* (NPIs). Contrasts like in (34) make it look like NPIs are sensitive to negation:

(34) a. * Every student has read *any* book(s).
b. No student has read *any* book(s).
c. * Some student has read *any* book(s).

But consider (35):

(35) a. Every student that has read *any* books thinks that they are valuable.
b. No student that has read *any* books thinks that they are valuable.
c. * Some student that has read *any* books thinks that they are valuable.

(36) NPIs are allowed in *downward-entailing* environments. (Ladusaw, 1979)

References

Barwise, Jon, and Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4:159–219.

Ladusaw, William A. 1979. Polarity sensitivity as inherent scope relations. Doctoral Dissertation, University of Texas at Austin.