

Sets, quantifiers, and entailment

1 Set notation and concepts

$\{0, 1\}$	a set with two members/elements, 0 and 1 We will often use 0 and 1 for the truth values, <i>false</i> and <i>true</i> .
$\{0, 1\} = \{1, 0\}$	Members of sets are unordered.
D_e	the <i>domain</i> of all individuals (in the model); sometimes also called the <i>universe</i>
\emptyset	the empty set, $\{\}$
$\{x : x \text{ is a cat}\}$	the set which contains everything in D_e that is a cat Could be read "the set of x such that x is a cat."
$x \in A$	x is a member of the set A Note that $a \in \{\{a\}, b\}$ is false.

We generally use uppercase letters for sets and lowercase letters for individuals in D_e .

- (1) $\text{Tama} \in \{x : x \text{ is a cat}\}$
- (2) $\{x : x \text{ saw a picture of } x\}$
- (3) $\{x : \text{this is semantics class}\}$

$A \cap B$	the <i>intersection</i> of A and B : $\{x : x \in A \text{ and } x \in B\}$
$A \cup B$	the <i>union</i> of A and B : $\{x : x \in A \text{ or } x \in B\}$
$B \setminus A$	the <i>complement</i> of A in B : $\{x : x \in B \text{ and } x \notin A\}$
$ A $	the <i>cardinality/size</i> of A ; for example $ \{0, 1\} = 2$

- (4) a. $\text{Tama} \in \{x : x \text{ is a cat}\} \cap \{y : y \text{ is black}\}$
b. $\text{Tama} \in \{x : x \text{ is a cat and } y \text{ is black}\}$
c. $\text{Tama} \in \{x : x \text{ is a cat and } x \text{ is black}\}$

$A \subseteq B$	A is a <i>subset</i> of B ; every member of A is in B .
$A = B$	$A \subseteq B$ and $B \subseteq A$

The sets above are all subsets of D_e , but we can also write sets of other things:

- (5) $\{A : A \subseteq \{x : x \text{ is a cat}\}\}$

What is in this set?

Some other notation, useful in proofs:

\Rightarrow	entails
\iff	if and only if (iff)

Exercises

(6) Prove that, for any sets A and B , $A \setminus B \subseteq A$. When is $A \setminus B = A$?

Proof: Suppose $x \in A \setminus B \iff x \in A$ and $x \notin B$
 $\implies x \in A$

Equality holds when A and B do not overlap: $A \cap B = \emptyset$

(7) Prove that, for any sets A and B , $A \cap B \subseteq A \cup B$. When is $A \cap B = A \cup B$?

Proof: Suppose $x \in A \cap B \iff x \in \{y : y \in A \text{ and } y \in B\}$
 $\iff x \in A$ and $x \in B$
 $\implies x \in A$ or $x \in B$
 $\iff x \in A \cup B$

Hints for proofs with sets:

- First, choose some arbitrary but concrete sets A and B and see if the statements are true and make sense to you.
- Make sure you consider $A = \emptyset$ and $B = \emptyset$ too.
- To prove that $A \subseteq B$, show that any member of A is necessarily also a member of B .
- To prove that $A = B$, show that $A \subseteq B$ and $B \subseteq A$.

2 Predicates as sets

- (8) a. Tama is a cat.
b. Tama is a black cat.

What is the meaning relationship between (8a) and (8b)? Is it an entailment or presupposition?

- (9) a. $A = \llbracket \text{cat} \rrbracket = \{x : x \text{ is a cat}\}$
b. $B = \llbracket \text{black cat} \rrbracket = \{x : x \text{ is black and } x \text{ is a cat}\}$

What is the relationship between A and B ?

NB: Winter's Truth Conditionality Criterion ensures the parallel between (8) and (9).

The TCC requires that S_1 intuitively entail S_2 iff, for any model M , $\llbracket S_1 \rrbracket^M \leq \llbracket S_2 \rrbracket^M$. Let the truth conditions of (8a,b) be:

- $\llbracket \text{Tama is a cat (8a)} \rrbracket^M = 1$ iff Tama $\in A$ in M
- $\llbracket \text{Tama is a black cat (8b)} \rrbracket^M = 1$ iff Tama $\in B$ in M

The fact that $(8b) \Rightarrow (8a)$ means that, for any model, if Tama $\in B$, Tama $\in A$. But this is equivalent to saying that in any model, $B \subseteq A$.

Exercises

What is the relationship between the (a) and (b) sets?

- (10) a. $\{x : x \text{ has read two books}\}$ (11) a. $\{x : x \text{ is a student that has read two books}\}$
b. $\{x : x \text{ has read three books}\}$ b. $\{x : x \text{ is a student that has read three books}\}$

3 Quantificational determiners

Consider quantifiers in subject position. What are the truth conditions of these sentences?

- (12) Every cat sleeps.
(13) Some cat sleeps.

Following an influential approach called *Generalized Quantifier Theory* (Barwise and Cooper, 1981), we will think about the meaning of quantifiers as relations between sets.

- (14) $\llbracket \text{Every cat sleeps} \rrbracket^M = 1$ iff $\llbracket \text{cat} \rrbracket^M \subseteq \llbracket \text{sleep} \rrbracket^M$
(15) $\llbracket \text{Some cat sleeps} \rrbracket^M = 1$ iff $\llbracket \text{cat} \rrbracket^M \cap \llbracket \text{sleep} \rrbracket^M \neq \emptyset$

Based on these results, we can define the general functions *every* and *some*:

- (16) $every(A)(B) = 1$ iff $A \subseteq B$ returns the truth value for “Every A is B.”
(17) $some(A)(B) = 1$ iff $A \cap B \neq \emptyset$ returns the truth value for “Some A is B.”

Exercise: Take a quantifier in a sentence of the form “Quantifier A B.” For example, in “Every student is sleeping,” the quantifier is “every,” A = “student,” and B = “is sleeping.” Define the truth conditions of “Quantifier A B” in terms of the sets A and B. (We ignore number morphology and agreement here.)

Answers:

- (18) $every(A)(B) = 1$ iff $A \subseteq B$
(19) $a/some(A)(B) = 1$ iff $A \cap B \neq \emptyset$
(20) $no(A)(B) = 1$ iff $A \cap B = \emptyset$
(21) $two(A)(B) = 1$ iff $|A \cap B| \geq 2$
(22) $more-than-two(A)(B) = 1$ iff $|A \cap B| > 2$
(23) $most(A)(B) = 1$ iff $|A \cap B| > |A \setminus B|$
(24) $not\ all(A)(B) = 1$ iff $A \setminus B \neq \emptyset$
(25) $the(A)(B) = 1$ iff $|A| = 1$ and $A \subseteq B$

4 Downward entailment and NPIs

What is the meaning relationship between these pairs, which vary in the underlined positions?

- (26) a. Stephanie read two books. (27) a. I know that S. reads two books.
b. Stephanie read three books. b. I know that S. reads three books.
- (28) a. I doubt that Stephanie read two books. (29) a. If S. reads two books, she will pass.
b. I doubt that Stephanie read three books. b. If S. reads three books, she will pass.
- (30) A function f is *downward-entailing* (DE; or downward monotone) if and only if for all $A \subseteq B \subseteq D_e$, $f(B) \Rightarrow f(A)$.

Now consider noun phrases like *any books*, *anyone*, *anything*. These are called *negative polarity items* (NPIs). Contrasts like in (40) make it look like NPIs are sensitive to negation:

- (31) a. *Stephanie read any book(s).
b. *I know that Stephanie read any book(s).
c. I doubt that Stephanie read any book(s).
d. If Stephanie read any book(s), she will pass.
- (32) NPIs are allowed in *downward-entailing* environments. (Ladusaw, 1979)

For quantificational determiners, we have to be clear whether we're looking at the first argument or second argument.

- (33) a. Every cat sleeps. (34) a. No cat sleeps. (35) a. Some cat sleeps.
b. Every black cat sleeps. b. No black cat sleeps. b. Some black cat sleeps.
- (36) a. Every student has read two books. (38) a. Some student has read two books.
b. Every student has read three books. b. Some student has read three books.
- (37) a. No student has read two books.
b. No student has read three books.
- (39) a. Every student that has read any book(s) will pass.
b. No student that has read any book(s) will pass.
c. *Some student that has read any book(s) will pass.
- (40) a. *Every student has read any book(s).
b. No student has read any book(s).
c. *Some student has read any book(s).

References

- Barwise, Jon, and Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4:159–219.
- Ladusaw, William A. 1979. Polarity sensitivity as inherent scope relations. Doctoral Dissertation, University of Texas at Austin.