Formal foundations and generalized quantifiers

Today we continue our goal of giving explicit *truth conditions* for meaning and we also begin to describe the meaning of some sub-sentential constituents.

1 Some concepts and notation

1.1 Sets

$\{0, 1\}$	a *set* with two *members/elements*, 0 and 1
	We will often use 0 and 1 for the truth values, *false* and *true*.
$\{0, 1\} = \{1, 0\}$	Members of sets are unordered.
D_e	the *domain* of all individuals; sometimes also called the *universe*
	This set may be restricted depending on the context we are discussing.
$\{x : x \text{ is a cat}\}$	the set which contains everything in D_e that is a cat
	Could be read "the set of x such that x is a cat."
$x \in A$	x is a member of the set A
	Note that $a \in \{\{a\}, b\}$ is false

We generally use uppercase letters for sets and lowercase letters for individuals.

(1) Kaline $\in \{x : x \text{ is a cat}\}$

(2) $\{x : x \text{ saw a picture of } x\}$

(3) $\{x : \text{this is semantics class}\}$

$A \cap B$	the *intersection* of A and B: $\{x : x \in A \text{ and } x \in B\}$
$A \cup B$	the *union* of A and B: $\{x : x \in A \text{ or } x \in B\}$
$B \setminus A$	the *complement* of A in B: $\{x : x \in B \text{ and } x \notin A\}$
$\|A\|$	the *cardinality/size* of A; for example $\|\{0, 1\}\| = 2$

(4) a. Kaline $\in \{x : x \text{ is a cat}\} \cap \{y : y \text{ is black}\}$
 b. Kaline $\in \{x : x \text{ is a cat and } y \text{ is black}\}$
 c. Kaline $\in \{x : x \text{ is a cat and } x \text{ is black}\}$

$A \subseteq B$	A is a *subset* of B; every member of A is in B.
$A = B$	$A \subseteq B$ and $B \subseteq A$

The sets above are all subsets of D_e, but we can also write sets of other things:

(5) $\{A : A \subseteq \{x : x \text{ is a cat}\}\}$

What is in this set?

Exercises

(6) Prove $A \setminus B \subseteq A$ (and when does equality hold?)

Proof: Suppose $x \in A \setminus B \iff x \in A$ and $x \notin B$

$\implies x \in A$

Equality holds when A and B do not overlap: $A \cap B = \emptyset$

(7) Prove $A \cap B \subseteq A \cup B$

Proof: Suppose $x \in A \cap B \iff x \in \{y : y \in A \text{ and } y \in B\}$

$\iff x \in A$ and $x \in B$

$\implies x \in A$ or $x \in B$

$\iff x \in A \cup B$

A note on technique: to prove that $A \subseteq B$, show that any member of A is necessarily also a member of B. To prove that $A = B$, show that $A \subseteq B$ and $B \subseteq A$.

1.2 Functions

We are familiar with functions from math: if $f(x) = x^2$, f takes a number and returns a number. The important property of a function is that it returns a *unique* value.

We can explicitly define a function by listing its inputs and outputs:

(8) $$g = \begin{bmatrix} 1 \to 2 \\ 2 \to 5 \\ 3 \to 5 \end{bmatrix}$$

Consider simple sentences about individual subjects, like "Ann sleeps." Given world knowledge about who sleeps, we could explicitly describe "sleep" as a function from individuals to truth values:

(9) $$[\![\text{sleep}]\!] = \begin{bmatrix} \text{Ann} \to 1 \\ \text{Jan} \to 1 \\ \text{Maria} \to 0 \end{bmatrix}$$

$[\![...]\!]$ the *denotation function*; it takes a linguistic expression and returns its meaning

1.3 Sets and functions

The set in (9) is a special kind of function: its outputs are in $\{0, 1\}$. This means we could also think of it as defining the set of people who sleep, with the outputs defining whether they are in the set or not.

(10) $[\![\text{sleep}]\!] = \{\text{Ann, Jan}\}$

Together with H&K, we will go back and forth between set and function notation.

1.4 Other notation

$\forall$	for all...
$\exists$	there exists...
$\wedge$	and
$\vee$	or
$\Rightarrow$	entails
$\iff$	if and only if (iff)

(11) $\forall x \in A, x \in B \iff A \subseteq B$

(12) $\exists x \in A, x \in B \iff A \cap B \neq \emptyset$

It is easy to think of situations where (11) is false and (12) is true, or situations where they are both true. Is there a situation where (11) is true but (12) is false?

2 Quantifiers

(13) Everyone sleeps.

(14) Someone sleeps.

What are these subjects "everyone" or "someone"? They are not in the domain of individuals D_e. Instead, the quantifiers are telling us something about the denotation of sleep.

(15) "Everyone sleeps":

a. A first approximation:
For every x in D_e (all relevant individuals), x sleeps.

b. In function terms:
$[\![\text{everyone sleeps}]\!] = 1$ iff $\forall x \in D_e$. $[\![\text{sleep}]\!](x) = 1$

c. In set terms:
$[\![\text{everyone sleeps}]\!] = 1$ iff $D_e \subseteq [\![\text{sleep}]\!]$

(16) "Someone sleeps":

a. A first approximation:
There exists a x in D_e, such that x sleeps.

b. In function terms:
$[\![\text{someone sleeps}]\!] = 1$ iff $\exists x \in D_e$. $[\![\text{sleep}]\!](x) = 1$

c. In set terms:
$[\![\text{someone sleeps}]\!] = 1$ iff $D_e \cap [\![\text{sleep}]\!] \neq \emptyset$

Following an influential approach called *Generalized Quantifier Theory* (Barwise and Cooper, 1981), we will think about the meaning of quantifiers as relations between sets. Based on the (c) results above, we can define the functions *everyone* and *someone* as follows:

(17) $everyone(B) = 1$ iff $D_e \subseteq B$ returns the truth value for "Everyone Bs."

(18) $someone(B) = 1$ iff $D_e \cap B \neq \emptyset$ returns the truth value for "Someone Bs."

Class exercise: Consider a quantifier in a sentence of the form "Quantifier A B." For example, in "Every student is sleeping," the quantifier is "every," A = "student," and B = "is sleeping." Define the truth conditions of "Quantifier A B" in terms of the sets A and B. (We ignore number morphology and agreement here.)

Answers:

(19) $every(A)(B) = 1$ iff $A \subseteq B$

(20) $a/some(A)(B) = 1$ iff $A \cap B \neq \emptyset$

(21) $no(A)(B) = 1$ iff $A \cap B = \emptyset$

(22) $two(A)(B) = 1$ iff $|A \cap B| = 2$

(23) $more\text{-}than\text{-}two(A)(B) = 1$ iff $|A \cap B| > 2$

(24) $most(A)(B) = 1$ iff $|A \cap B| > |A \setminus B|$

(25) $not\ all(A)(B) = 1$ iff $A \setminus B \neq \emptyset$

(26) $the(A)(B) = 1$ iff $|A| = 1$ and $A \subseteq B$

References

Barwise, Jon, and Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4:159–219.